# Numerical Solution of Two-dimensional Heat-flow Problems

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Two-dimensional heat flow frequently leads to problems not amenable to the methods of classical mathematical physics; thus, procedures for obtaining approximate solutions are desirable. A recently introduced finite-difference method, known to be applicable to problems in a rectangular region and involving much less calculation than previous methods, is extended by example to cases of more practical interest. Although all three examples given are steady state, unsteady state problems may also be attacked successfully by the method. The first example is that of flow around a corner and indicates that a more complicated region than a rectangle can be treated. Then a problem involving a radiation-boundary condition is given; as this condition is nonlinear, the method is extended to more general equations. The last example involves point heat sources and sinks in an elliptical region and so extends the method to treat curved boundaries (as distinguished from polygonal domains) and singular points. It is believed that materially less calculation is necessary by this method than for previous procedures.

Many cases of heat conduction in three dimension may be reduced by symmetry to problems involving only two-space variables. Further reduction to a one-space variable is frequently not possible. Unless the geometry and boundary conditions are very simple, analytical solutions of the differential equation of heat conduction in two dimensions are either very cumbersome or impossible to obtain. Approximate solutions may be obtained by numerical, graphical, or experimental methods. The rapid development of high-speed electronic computers in the past few years has stimulated interest in numerical methods for the solution of a wide class of problems. In this paper a recently introduced numerical procedure which is known to be applicable to problems in a rectangular region is extended by example to cases of more practical interest. Although this procedure was developed for the purpose of solving problems involving fluid flow through porous media in two-dimensional reservoirs, it can also be used for two-dimensional problems in many other fields such as heat conduction and diffusion.

## PREVIOUS NUMERICAL METHODS

Heat-conduction problems can be divided into two categories: unsteady state and steady state. The differential equation describing unsteady state heat flow is (*8*)

$$\frac{k}{c\rho}\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)=\frac{\partial T}{\partial \theta} \qquad (1)$$

while that governing steady state heat flow is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \qquad (2)$$

All numerical methods for the solution of these equations involve replacement of the continuous derivatives by ratios of finite differences and solution of the resulting difference equations. To formulate the difference equations, the two-dimensional region in which the integration is to be carried out has placed over it an integration net with mesh widths $\Delta x$ and $\Delta y$, as shown in Figure 1. Usually

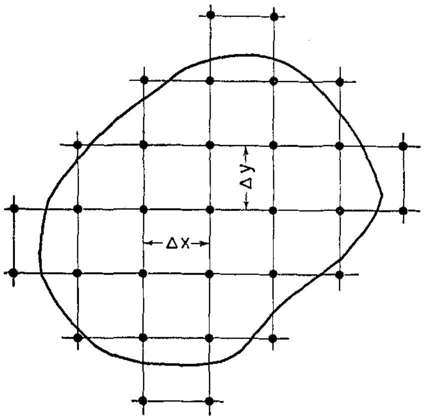

**Fig. 1. Integration net.**

$\Delta x$ and $\Delta y$ are set equal to each other. The second derivatives are then replaced by second differences. For example, at the point $x$, $y$, $\partial^2 T/\partial x^2$ is replaced by

$$\frac{\dfrac{T(x+\Delta x)-T(x)}{\Delta x}-\dfrac{T(x)-T(x-\Delta x)}{\Delta x}}{\Delta x}$$

which is equal to

$$\frac{1}{(\Delta x)^2}[T(x-\Delta x)-2T(x)+T(x+\Delta x)]$$

Thus, if $\Delta x=\Delta y$, the difference equation corresponding to the steady state Equation (2) is

$$T(x-\Delta x,y)+T(x+\Delta x,y)+T(x,y-\Delta y)+T(x,y+\Delta y)-4\,T(x,y)=0 \qquad (3)$$

In the carrying out of a numerical solution of the unsteady state Equation (1), the time coordinate is also divided into finite increments, $\Delta\theta$, not necessarily equal. It is assumed that a solution has been obtained for time $\theta$, and a difference equation is used to obtain the solution at time $\theta+\Delta\theta$.

Jakob (*4*) and Milne (*5*) describe several methods for the numerical solution of Equations (1) and (2). The three chief ones are relaxation, explicit, and implicit methods. Relaxation is rapid and convenient for hand calculation with a small number of points, but when it is desired to evaluate the solution at a large number of points by use of a computing machine, relaxation is inconvenient because of its non-mechanical nature.

The terms *explicit* and *implicit* have somewhat different meanings for the unsteady and steady state cases. In the unsteady state case, an explicit difference equation may be solved explicitly for the unknown function at time $\theta + \Delta\theta$ in terms of the known values of the function at time $\theta$. For example, an explicit difference equation may be obtained by replacing the second derivatives of Equation (1) by second differences evaluated at time $\theta$. Thus, if $\Delta x = \Delta y$,

$$\frac{k}{c\rho(\Delta x)^2}[T(x - \Delta x, y, \theta) + T(x + \Delta x, y, \theta) + T(x, y - \Delta y, \theta) + T(x, y + \Delta y, \theta) - 4\,T(x, y, \theta)] = \frac{1}{\Delta\theta}[T(x, y, \theta + \Delta\theta) - T(x, y, \theta)] \quad (4)$$

This equation may be solved explicitly for $T(x, y, \theta + \Delta\theta)$ at each point of the integration net. Starting with known initial values of $T$ at $\theta = 0$, the solution may be carried forward to any time, $\theta$.

Implicit methods for the unsteady state case, on the other hand, are characterized by having the unknown values of the desired function at time $\theta + \Delta\theta$ bound together by systems of simultaneous equations. For example, an implicit difference equation may be obtained by replacing the second derivatives of Equation (1) by second differences evaluated at time $\theta + \theta\Delta$. Thus, if $\Delta x = \Delta y$.

$$\frac{k}{c\rho(\Delta x)^2}[T(x - \Delta x, y, \theta + \Delta\theta) + T(x + \Delta x, y, \theta + \Delta\theta) + T(x, y - \Delta y, \theta + \Delta\theta) + T(x, y + \Delta y, \theta + \Delta\theta) - 4\,T(x, y, \theta + \Delta\theta)] = \frac{1}{\Delta\theta}[T(x, y, \theta + \Delta\theta) - T(x, y, \theta)] \quad (5)$$

When this equation is written for each point in the integration net, $N$ simultaneous equations are obtained, where $N$ is the number of interior points in the net. These $N$ equations must be solved at each time step.

When the steady state case is considered, the difference Equation (3) is written for each point in the integration net. A set of $N$ simultaneous equations is obtained, most of which have five unknowns in them. Simultaneous equations of this type are most successfully attacked by iterative methods, most of which can be shown to be analogous to a solution by finite differences of a related unsteady state problem. An iteration method is explicit if each iteration involves solving explicitly for the unknown function at each point, while it is implicit if the unknown values of the function are tied together by simultaneous equations.

For the solution of the unsteady state problem on a computing machine, the explicit Equation (4) is easy to use because of its simplicity, but it suffers from a severe disadvantage. If the size of the time increment, $\Delta\theta$, is taken too large, the computations become unstable in the sense that small errors introduced at the beginning of the calculations are amplified and eventually become so large as to swamp the calculations. To avoid this, it is necessary to satisfy the relationship

$$\Delta\theta \leqslant \frac{c\rho\,(\Delta x)^2}{4k} \quad (6)$$

This instability of explicit difference equations for large time increments is discussed by O'Brien, Hyman and Kaplan (*6*) and Milne (*5*). This restriction on $\Delta\theta$ means that a large number of small time steps must be taken to carry the solution to a given time $\theta$. If it is desired to increase the accuracy of the solution by using more points in the integration net, it is necessary to decrease $\Delta\theta$ in accordance with Equation (6), so that the number of computations increases as the square of the number of points.

The implicit difference Equation (5), on the other hand, has the advantage of being stable for any size time increment (*1, 6*). In the carrying out of an unsteady state solution by an implicit procedure, $\Delta\theta$ is restricted not by stability but only by the allowable truncation error, and for a specified accuracy, fewer time steps are required than for the explicit procedure. The reduction of the number of time steps more than compensates for the additional labor required to solve by iteration the $N$ simultaneous equations at each time step, so that, from the point of view of computational labor, the implicit procedure is considerably superior to the explicit procedure for the solution of unsteady state problems (*7*).

For the solution of the steady state problem on computing machines, various explicit methods have been proposed for iterating the simultaneous equations which occur when Equation (3) is written for each point in the integration net. Probably the best explicit iterative method known is the extrapolated Liebmann method (also known as the successive overrelaxation method). The computational labor required for this method is discussed by Frankel (*3*). No implicit methods for iterating simultaneous equations have previously been proposed.

## ALTERNATING-DIRECTION IMPLICIT METHOD

### Application to Unsteady State Problem

Peaceman and Rachford (*7*) proposed an alternating-direction implicit procedure for the solution of Equation (1) which is stable for any size of time increment and yet avoids the necessity for iteration at each time step. The difference equations are obtained by replacing only one of the second derivatives, say $\partial^2T/\partial x^2$, by a second difference evaluated in terms of the unknown values of $T$ at time $\theta + \Delta\theta$, while replacing the other second derivative, $\partial^2T/\partial y^2$, by a second difference evaluated in terms of known values of $T$ at time $\theta$. Comparatively small sets of simultaneous equations, each containing no more than three unknowns, are formed which are said to be implicit in the $x$ direction. These equations are of a form that can be solved easily without iteration. If the procedure is then repeated for a second time step of equal size, with the difference equations implicit in the $y$ direction, the over-all procedure for the two time steps is stable for any size of time increment.

Thus two difference equations are used, one for the first time step, the other for the second time step:

$$\frac{k}{c\rho\,(\Delta x)^2}[T(x - \Delta x, y, \theta + \Delta\theta) - 2T(x, y, \theta + \Delta\theta) + T(x + \Delta x, y, \theta + \Delta\theta)] + \frac{k}{c\rho\,(\Delta y)^2}[T(x, y - \Delta y, \theta) - 2\,T(x, y, \theta) + T(x, y + \Delta y, \theta)] = \frac{1}{\Delta\theta}[T(x, y, \theta + \Delta\theta) - T(x, y, \theta)] \quad (7)$$

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**ERRATA**

**Volume 1, Number 1, page 25:** The values of $k$ in Table 1 should be higher by a factor of 10.

**Volume 1, Number 1, page 76:** In the legend of Figure 1 $p_r = p^b{}_o/R(\varepsilon/k)$ should be $p_r = pb_o/R(\varepsilon/k)$.

**Volume 1, Number 2, page 146:** Equation (26) should read $(d^3P/dV^3)_T \leqslant 0$ at critical.

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$$-\frac{k}{c\rho(\Delta x)^2}[T(x-\Delta x, y, \theta+\Delta\theta) - 2T(x, y, \theta+\Delta\theta) + T(x+\Delta x, y, \theta+\Delta\theta)] + \frac{k}{c\rho(\Delta y)^2}\cdot[T(x, y-\Delta y, \theta+2\Delta\theta) - 2T(x, y, \theta+2\Delta\theta) + T(x, y+\Delta y, \theta+2\Delta\theta)] = \frac{1}{\Delta\theta}[T(x, y, \theta+2\Delta\theta) - T(x, y, \theta+\Delta\theta)] \quad (8)$$

If $\Delta x = \Delta y$, and $\alpha = c\rho(\Delta x)^2/k\Delta\theta$, these equations may be arranged in the following form, which is more suitable for calculation:

$$-T(x-\Delta x, y, \theta+\Delta\theta) + [2+\alpha]T(x, y, \theta+\Delta\theta) - T(x+\Delta x, y, \theta+\Delta\theta) = T(x, y-\Delta y, \theta) + [\alpha-2]T(x, y, \theta) + T(x, y+\Delta y, \theta) \quad (9)$$

$$-T(x, y-\Delta y, \theta+2\Delta\theta) + [2+\alpha]T(x, y, \theta+2\Delta\theta) - T(x, y+\Delta y, \theta+2\Delta\theta) = T(x-\Delta x, y, \theta+\Delta\theta) + [\alpha-2]T(x, y, \theta+\Delta\theta) + T(x+\Delta x, y, \theta+\Delta\theta) \quad (10)$$

Use of Equation (9) or (10) for each point in the integration net leads to as many sets of simultaneous equations as there are lines in the net, each set having as many equations and unknowns as there are points on the line. When the boundary conditions are taken into account, these equations may always be arranged in the form

$$B_1 T_1 + C_1 T_2 = D_1$$

$$A_i T_{i-1} + B_i T_i + C_i T_{i+1} = D_i \qquad 2 \leqslant i \leqslant n-1$$

$$A_n T_{n-1} + B_n T_n = D_n \quad (11)$$

The solution of these equations is obtained by a straightforward technique proposed by L. H. Thomas of the Watson Scientific Computing Laboratory. Let

$$w_1 = B_1$$

$$w_i = B_i - \frac{A_i C_{i-1}}{w_{i-1}} \quad 2 \leqslant i \leqslant n \quad (12)$$

and

$$g_1 = \frac{D_1}{w_1}$$

$$g_i = \frac{D_i - A_i g_{i-1}}{w_i} \quad 2 \leqslant i \leqslant n \quad (13)$$

The solution is

$$T_n = g_n$$

$$T_i = g_i - \frac{C_i T_{i+1}}{w_i} \quad 1 \leqslant i \leqslant n-1 \quad (14)$$

Thus $w$ and $g$ are computed in order of increasing $i$, and $T$ is computed in order of decreasing $i$.

For the integration of Equation (1) in a square by this method, approximately $9N$ operations must be performed for each time step (*7*). For the complete solution with $N = 196$ the work required by the alternating-direction implicit method was found to be 1/7 of the work required by the use of the implicit Equation (5) with extrapolated Liebmann iteration, and 1/25 of the work required by the use of the explicit Equation (4). The comparison is even more favorable for the alternating-direction implicit method for larger values of $N$. Douglas(*1*) has shown that as the mesh width is decreased, the solution obtained converges to the solution of the differential equation.

#### Application to Steady State Problem

The alternating-direction implicit method may be used to iterate to the solution of the steady state Equation (2), since the steady state solution may be regarded as the limiting case of the solution of a corresponding unsteady state problem. In this case each stage of the iteration may be regarded as a time step of the unsteady state problem, while the starting values used for the first iteration correspond to the initial condition. Equations (9) and (10) are used for alternate stages of the iteration with $\alpha$ serving as an iteration parameter. For the solution of Equation (2) in a square, it was found(*7*) that a sequence of $\alpha$'s can be chosen such that this method requires about $\ln N/N^{1/2}$ as many calculations as the extrapolated Liebmann method.

#### Examples

The alternating-direction implicit procedure has been applied(*7*) in the solution of two simple problems involving square boundaries. The first problem considered was that of unsteady state heat flow in a square wherein the boundaries are maintained at zero temperature and the square initially has a temperature of unity. The second problem was that of determining the steady state temperature distribution in a square in which two opposite faces are at zero temperature and the remaining two faces have a temperature of unity.

It has been shown(*1, 7*) that the alternating-direction implicit procedure is stable for the case of a square or rectangular boundary, but stability has not been demonstrated for a more general class of boundaries. It is the purpose of this paper to show by example the application of the new procedure to problems involving more complicated boundaries. Three examples will be cited, all of which were calculated by means of eight-digit, floating-decimal arithmetic. The examples were so chosen as to keep the number of points in the integration nets small in order not to overrun the small internal storage capacity of the computer used. Although this limitation did prevent the examples from being more practical than they were, they are sufficient to show that a wide variety of boundary conditions can be treated satisfactorily by the new procedure.

All the examples discussed are steady state problems; however, the difference equations are set up as if the problems are unsteady state in nature. Thus it is shown that as far as setting up the equations is concerned, there is no difference between steady and unsteady state problems. The only difference is in the choice of the parameter, $\alpha$, and the discussion of this choice will be deferred to the next section.

*Flow Around a Corner.* The problem of flow of heat around a corner was chosen in order to demonstrate that a more complicated region than a rectangle can be treated. The equation describing this process also describes the flow of an ideal fluid around a corner; hence this application of the alternating-direction implicit procedure should be of interest in hydrodynamics.

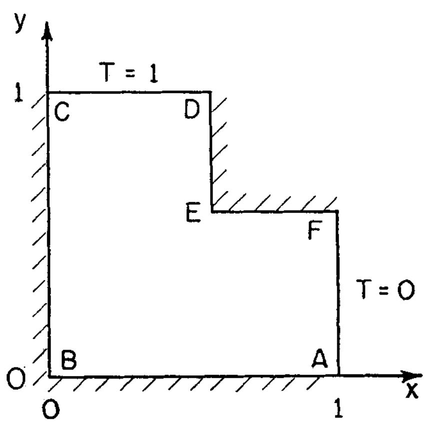


Fig. 2. Flow around corner.

In the shape in Figure 2 edge $AF$ is maintained at zero temperature; edge $CD$ is maintained at a temperature of unity; at all other edges no heat flows across the boundary. Equation (2) applies to the interior of this region, and the following boundary conditions apply:

$$\text{Edge } AF \qquad T = 0$$

$$\text{Edges } AB \text{ and } EF \; \frac{\partial T}{\partial y} = 0 \qquad (15)$$

$$\text{Edges } BC \text{ and } DE \; \frac{\partial T}{\partial x} = 0$$

$$\text{Edge } CD \qquad T = 1$$

Figure 3 shows the integration net with each of the points labeled with an index number.

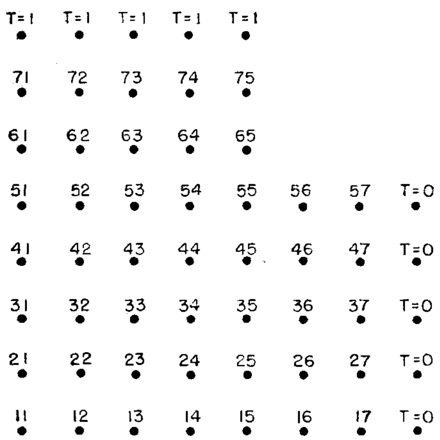


Fig. 3. Integration net for flow around corner.

First one considers the time step in which the difference equations are implicit in the $x$ direction. Equation (9) applies to each of the interior points. At the boundaries which are adiabatic, the temperature gradient in the direction perpendicular to the boundaries is zero, and the so-called "reflection" boundary condition can be used; that is, a set of fictitious points are imagined to exist, $\Delta x$, outside the boundary, and these points are assumed to have the same values as the points located a distance $\Delta x$ inside the boundary. Figure 4 shows how this applies to some of the points on edge $BC$. When Equation (9) is written for a point on $BC$, only two unknowns appear, and so this becomes the first equation for that particular line. When Equation (9) is written for one of the points just inside edge $AF$, again only two unknowns appear, and the last equation for that line

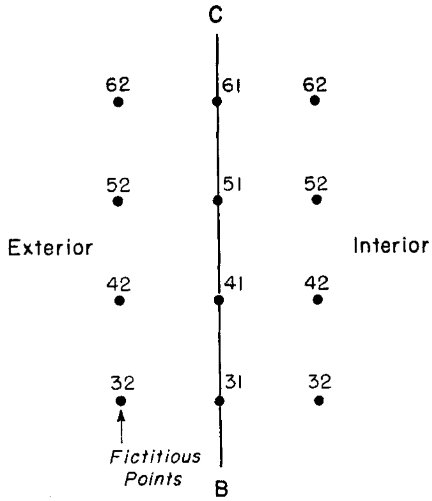


Fig. 4 Reflection boundary condition.

is obtained. Below are written the equations for the first, second, sixth, and seventh lines to illustrate how all the equations are set up. $T$ represents a known value of temperature at time $\theta$; $T^*$ represents an unknown value of temperature at time $\theta + \Delta\theta$; the subscripts are the index numbers of the points.

*First Line*

$$(2+\alpha)\,T^*_{11} - 2\,T^*_{12} = 2\,T_{21} + (\alpha-2)\,T_{11}$$

$$-T^*_{11} + (2+\alpha)\,T^*_{12} - T^*_{13} = 2\,T_{22} + (\alpha-2)\,T_{12}$$

$$-T^*_{12} + (2+\alpha)\,T^*_{13} - T^*_{14} = 2\,T_{23} + (\alpha-2)\,T_{13}$$

$$-T^*_{15} + (2+\alpha)\,T^*_{16} - T^*_{17} = 2\,T_{26} + (\alpha-2)\,T_{16}$$

$$-T^*_{16} + (2+\alpha)\,T^*_{17} - 0 = 2\,T_{27} + (\alpha-2)\,T_{17}$$

These seven equations with seven unknowns may now be solved for the unknown temperatures on the first line by use of Equations (12) to (14).

*Second Line*

$$(2+\alpha)\,T^*_{21} - 2\,T^*_{22} = T_{11} + (\alpha-2)\,T_{21} + T_{31}$$

$$-T^*_{21} + (2+\alpha)\,T^*_{22} - T^*_{23} = T_{12} + (\alpha-2)\,T_{22} + T_{32} - T^*_{25} + (2+\alpha)\,T^*_{26} - T^*_{27} = T_{16} + (\alpha-2)\,T_{26} + T_{36}$$

$$-T^*_{26} + (2+\alpha)\,T^*_{27} - 0 = T_{17} + (\alpha-2)\,T_{27} + T_{37}$$

*Sixth Line*

$$(2+\alpha)\,T^*_{61} - 2\,T^*_{62} = T_{51} + (\alpha-2)\,T_{61} + T_{71}$$

$$-T^*_{61} + (2+\alpha)\,T^*_{62} - T^*_{63} = T_{52} + (\alpha-2)\,T_{62} + T_{72} - 2\,T^*_{64} + (2+\alpha)\,T^*_{65} = T_{55} + (\alpha-2)\,T_{65} + T_{75}$$

*Seventh Line*

$$(2+\alpha)\,T^*_{71} - 2\,T^*_{72} = T_{61} + (\alpha-2)\,T_{71} + 1$$

$$-T^*_{71} + (2+\alpha)\,T^*_{72} - T^*_{73} = T_{62} + (\alpha-2)\,T_{72} + 1 - 2\,T^*_{74} + (2+\alpha)\,T^*_{75} = T_{65} + (\alpha-2)\,T_{75} + 1$$

Thus, line-by-line, the values of $T$ at $\theta + \Delta\theta$ are obtained. For the next time step, the difference equations are implicit in the $y$ direction and the same value of $\alpha$ is used. Equation (10) is now used for each of the interior points, and the equations are set up in exactly the same way as on the previous step. $T$ now represents a known value of temperature at time $\theta + \Delta\theta$; $T^*$ represents an unknown value at time $\theta + 2\theta\Delta$. Just the first line will be written.

*First Line*

$$(2+\alpha)\,T^*_{11} - 2\,T^*_{21} = 2\,T_{12} + (\alpha-2)\,T_{11}$$

$$-T^*_{11} + (2+\alpha)\,T^*_{21} - T^*_{31} = 2\,T_{22} + (\alpha-2)\,T_{21}$$

$$-T^*_{21} + (2+\alpha)\,T^*_{31} - T^*_{41} = 2\,T_{32} + (\alpha-2)\,T_{31}$$

$$-T^*_{61} + (2+\alpha)\,T^*_{71} - 1 = 2\,T_{72} + (\alpha-2)\,T_{71}$$

The values of $T$ at $\theta + 2\Delta\theta$ are obtained, again line by line. Then then entire procedure is repeated, first the equation implicit in the $x$ direction being used, then the equations implicit in the $y$ direction. A different value of $\alpha$ may be used for each of the double steps, but it must be the same for both halves of the double step.

By use of these equations, the steady state solution was obtained by iteration; the starting values, by use of the arbitrary equation

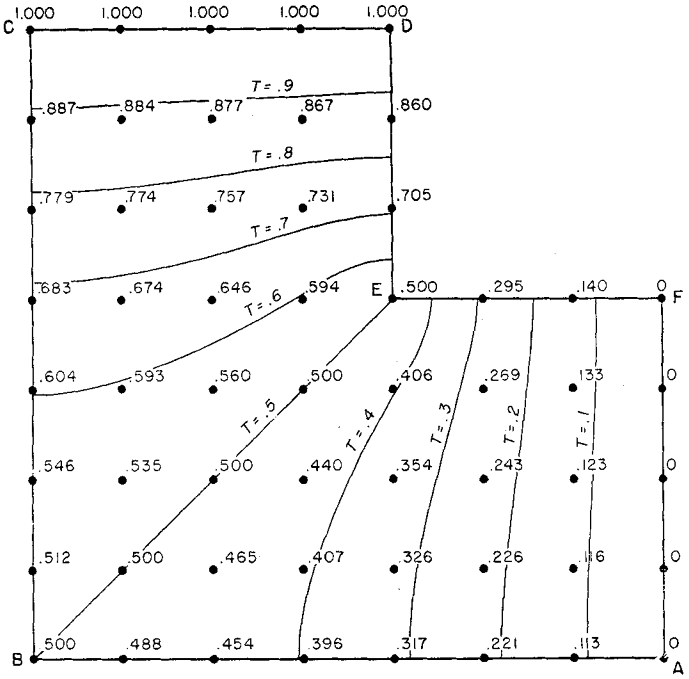


Fig. 5. Temperature distribution for heat flow around corner.

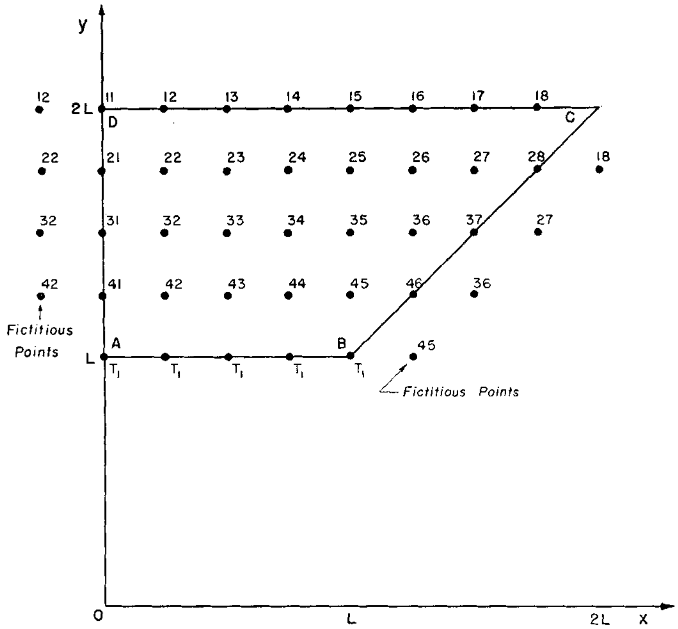


Fig. 6. Integration net for radiation from square pipe.

$$T = 0.5\,(1-x)/(1-y)$$

for all points to the right of the line $x = y$, and the equation

$$T = 1 - 0.5\,(1-y)/(1-x)$$

for all points above the line $x = y$. The values of $\alpha$ used were 4, 2, 1, 0.5, 4, 2, 1, 0.5 . . . with the cycle of four values of $\alpha$ being repeated.

To check the progress of the iteration after each double step, the residual, $\delta$, was calculated for each interior point by the equation

$$\delta = T(x - \Delta x, y) + T(x + \Delta x, y) + T(x, y - \Delta y) + T(x, y + \Delta y) - 4\,T(x, y) \qquad (16)$$

and the sum of the squares of the residuals, $\Sigma\delta^2$, evaluated. A plot of log $\Sigma\delta^2$ vs. number of double steps fell approximately on a straight line for the first nine double steps, with $\Sigma\delta^2$ decreasing from 0.187 to $1.04 \times 10^{-9}$. Twelve double steps reduced $\Sigma\delta^2$ to $2 \times 10^{-10}$, corresponding to about five digits' accuracy. Upon further iterations, the solution continued to converge, but at a much slower rate, with $\Sigma\delta^2$ reduced to $10^{-12}$ after twenty-eight double steps, corresponding to about six digits' accuracy. The solution is shown in Figure 5. The isotherms in this figure were obtained by graphical interpolation between the points.

*Radiation from Square Pipe.* The second example is that of determining the temperature distribution in a square pipe, the inside surface of which is maintained at a constant temperature, $t_1$, while the outside surface is radiating heat to surroundings maintained at temperature $t_2$. This problem is of interest because it contains a nonlinear boundary condition involving radiation and also because it introduces a diagonal boundary. The ability of the new numerical procedure to handle nonlinear boundary conditions is important for furnace-design problems, as well as for problems with heat transfer to fluids in which the heat transfer coefficient is not constant.

Because of symmetry, only one eighth of the cross section of the pipe need be considered. By this use of symmetry, it is possible to calculate the temperature distribution in the whole pipe with many fewer points. The integration net used is shown in Figure 6. The bore of the pipe is assumed to be a square $2L$ across, the outside a square $4L$ across.

The equation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0 \qquad (17)$$

holds in the interior of the region, and the boundary conditions are

$$\text{Edge } AB: \quad t = t_1$$

$$\text{Edge } BC: \quad \partial t/\partial n = 0$$

where $n$ is distance in a direction perpendicular to the edge

$$\text{Edge } CD: \quad -k\frac{\partial t}{\partial y} = \sigma\,(t^4 - t_2^4)$$

$$\text{Edge } AD: \quad \frac{\partial t}{\partial x} = 0 \qquad (18)$$

Because of the symmetry, edges $BC$ and $AD$ have reflection boundary conditions. The fictitious points associated with these edges are also shown in Figure 6.

In order to make the functions dimensionless, the following substitutions may be made:

$$X = x/L \qquad (19)$$

$$Y = y/L \qquad (20)$$

$$T = (\sigma L/k)^{1/3}\,t \qquad (21)$$

Equation (17) becomes

$$\frac{\partial^2 T}{\partial X^2} + \frac{\partial^2 T}{\partial Y^2} = 0 \qquad (22)$$

and the upper and lower boundary conditions become

$$\text{Edge } AB: \quad T = T_1 = (\sigma/kL)^{1/3}\,t_1 \qquad (23)$$

$$\text{Edge } CD: \quad -\frac{\partial T}{\partial Y} = T^4 - T_2^4 \qquad (24)$$

In the setting up of the difference equations, no new difficulties are introduced at the boundaries $AB$

and $AD$. When the difference equations are set up for the points on the diagonal boundary, the temperature at the fictitious points are introduced as unknowns. If the calculations are carried out so that the long lines are solved before the short ones, it always turns out that the temperature at each fictitious point has been obtained during the solution of the previous line. Thus only two unknowns appear in the equation for the diagonal boundary point, and this equation becomes the last one for the corresponding line.

The difference Equations (9) and (10) are not written for the points on the boundary $CD$. Instead, a difference equation corresponding to Equation (24) is written. In order that this difference equation be linear in $T$, the $T^4$ term is split into two factors, $T^3\,T^*$, where $T$ is the known value of temperature obtained from the last time step and $T^*$ is the unknown value. The boundary difference equation is then

$$\frac{T^*_{2i}-T^*_{1i}}{\Delta Y} = T^3_{1i}\,T^*_{1i} - T_2$$

or

$$-T^*_{2i} + (1 + T^3_{1i}\Delta Y)\,T^*_{1i} = T_2^4\,\Delta Y \qquad (25)$$

Consider first the difference equations implicit in the $x$ direction. Line 21-28 is solved first, by use of the following equations.

*Second Line*

$$(2+\alpha)\,T^*_{21} - 2\,T^*_{22} = T_{11} + (\alpha - 2)\,T_{21} + T_{31}$$

$$-T^*_{26} + (2+\alpha)\,T^*_{27} - T^*_{28} = T_{17} + (\alpha - 2)\,T_{27} + T_{37}$$

$$-T^*_{27} + (2+\alpha)\,T^*_{28} - T^*_{18} = T_{18} + (\alpha - 2)\,T_{28} + T_{27}$$

$$-T^*_{28} + (1 + T^3_{18}\,\Delta Y)\,T^*_{18} = T_2^4\,\Delta Y$$

In addition to $T^*_{21}$ through $T^*_{28}$, $T^*_{18}$ is obtained. By use of Equation (25) $T^*_{11}$ through $T^*_{17}$ may be solved for directly. The remaining lines offer no difficulty.

In the setting up of the difference equations implicit in the $y$ direction, Equation (25) is used as the first equation for each line. Otherwise the equations for the second half of the double time step are quite straightforward.

In order to carry out a numerical solution of these equations, values were assigned to the constants $T_1$ and $T_2$. It was assumed that $L = 4$ in., $k = 30$ B.t.u./(hr.)(ft.)(°F.), $t_1 = 1033$°F. $= 1493$°R., and $t_2 = 100$°F. $= 560$°R. $\sigma$ is equal to $0.173 \times 10^{-8}$ B.t.u./(sq.ft.)(hr.)(°R.)$^4$. Then $T_1 = 0.4$ and $T_2 = 0.15$. $\Delta Y$ is equal to 0.25.

The starting values for the iteration were obtained by solving for the temperature distribution that would exist in an infinite plate of thickness $L$ with the same upper and lower boundary conditions. This gave at the upper edge $T = 0.37971694$. Intermediate starting temperatures were obtained by use of the relation

$$T = 0.4 - (0.4 - 0.37971694)\,(Y - 1)$$

Residuals for the interior points were calculated by Equation (16), and the sum of squares of the residuals were evaluated after each double step. The values of $\alpha$ used were 4, 2, 1, 0.5, 0.2, 4, 2, 1, 0.5, 0.2, 10, 4, 10, 20, 10, 4, 2, 1, 0.5, 0.2, 0.1, 20, 10. A plot of log $\Sigma\delta^2$ vs. number of steps was approximately straight for the first twelve double steps, with $\Sigma\delta^2$ decreasing from $3.06 \times 10^{-4}$ to $7.66 \times 10^{-11}$; the portion of the plot from twelve to twenty-three double steps was also approximately straight, with $\Sigma\delta^2$ decreasing to $1.51 \times 10^{-13}$. The solution, with temperatures in degrees Fahrenheit, is presented in Figure 7.

*Heat Sources and Sinks in an Elliptical Plate.* Although the examples discussed above have straight boundaries, the boundaries of many problems that occur in engineering are curved. In addition, some problems involve point sources and sinks, wherein heat or fluid is added or removed at a point. These two features are introduced in the third example, which is that of determining the temperature distribution in an elliptical plate which has two heat sources and two heat sinks. (*See* Figure 8.)

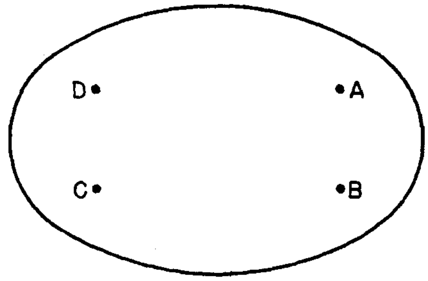


Fig. 8. Heat sources and sinks in an elliptical plate.

The elliptical boundary is adiabatic, and heat is injected at equal constant rates at points $A$ and $B$ and removed at equal constant rates from points $C$ and $D$.

This example has been formulated as a heat-conduction problem, but it may also be formulated as an equivalent electrical problem, where current flows into the plate at points $A$ and $B$ and flows out at points $C$ and $D$, and it is desired to compute the potential distribution. Another important analogy occurs in reservoir engineering, wherein it is desired to obtain the pressure distribution in an elliptical reservoir in which fluid is injected at wells located at points $A$ and $B$ and produced from wells located at points $C$ and $D$. Because well diameters are usually very small compared with the dimensions of a reservoir, they must be considered as points. This example shows the application of the new method to reservoir engineering, since reservoirs almost always have curved boundaries and have many wells located at various points within.

The problem as stated above has only derivatives specified at the boundaries, and so the solution is unique except for an arbitrary con-

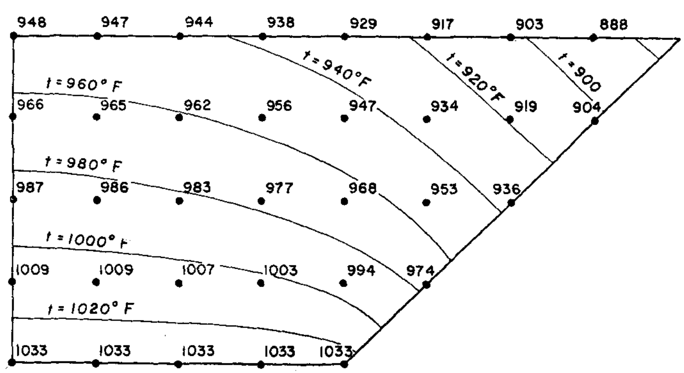


Fig. 7. Temperature distribution in a square pipe with radiation from outer surface.

stant. From symmetry it can be seen that the temperature will be constant along the minor axis; a zero value will be arbitrarily assigned to this constant temperature. The solution will be symmetric with respect to $y$ and antisymmetric with respect to $x$, and so it will be sufficient to consider only the upper right-hand quadrant. Figure 9 shows the integration net used. Since the major axis is now a reflection boundary, fictitious points have been added below the major axis. Fictitious points have also been added outside the curved boundary in such a way that every interior point is surrounded by four mesh points.

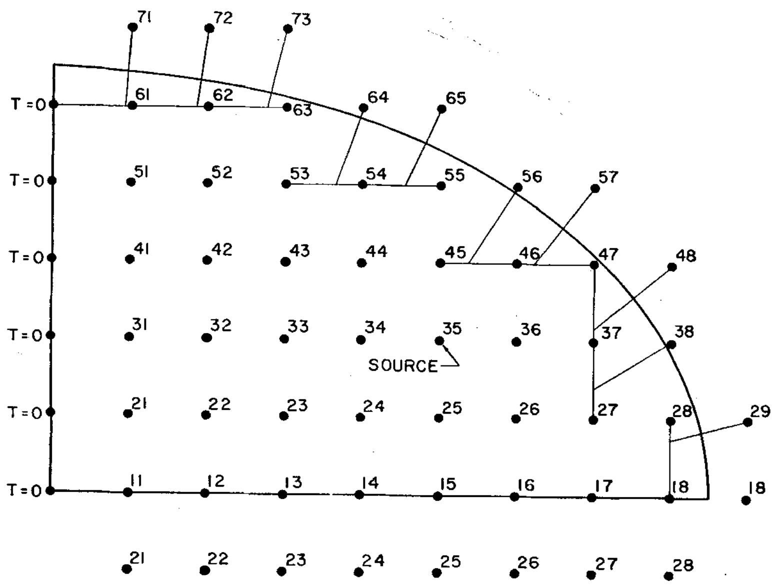


Fig. 9. Integration net for elliptical plate problem.

The problem now exists of evaluating the temperature at the fictitious points outside a curved boundary at which the normal derivative is zero. The device proposed by Fox(*2*), who was concerned with applying the relaxation method to the solution of problems with curved boundaries at which the normal derivative is specified, is applicable here. The device is to draw a line from each fictitious point perpendicular to the curved boundary and to extend it until it intersects a line which joins two mesh points completely within the boundary. This has been done in Figure 9. Since the normal derivative of the temperature is specified to be zero at the boundary, the temperatures at both ends of the perpendicular lines are taken to be equal. Linear interpolation along the line joining the two mesh points within the boundary is then used to evaluate this temperature. For example, the perpendicular from point 73 intersects the line approximately one-quarter of the way from point 63 to point 62. Then one writes

$$T_{73} = 0.25\, T_{62} + 0.75\, T_{63} \quad (26)$$

When Equation (9) or (10) is written about a point just inside the boundary, a fictitious point will be involved on either the left- or right-hand side of the difference equation. If it is on the right side, it is necessary to use the old values of the two interior temperatures used to evaluate the fictitious point. For example, if $T_{73}$ occurs on the right side, Equation (26) is used. On the other hand, if the fictitious temperature occurs on the left-hand side of the difference equation, new or unknown values of the interior temperatures must be used. Thus, if $T^*_{73}$ occurs on the left side, one must write

$$T^*_{73} = 0.25\, T^*_{62} + 0.75\, T^*_{63} \quad (27)$$

It frequently occurs that an unknown value of $T^*$ from a different line will be introduced by such a difference equation. It will always be from a longer line, however, and so if the lines are solved in order of decreasing length, this $T^*$ will have just been solved for on the previous line.

It remains now to take into account the point source occurring at point 35. While the final solution satisfies Equation (3) everywhere else in the interior of the region, it does not satisfy it at a point source or sink because this equation expresses the condition that the net heat flow into the region around a point is zero. To obtain the correct expression, one must consider Figure 10.

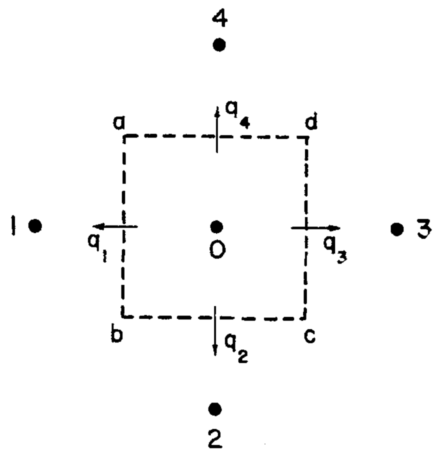


Fig. 10. Heat source at a point.

If heat is injected at point 0 at the rate $q$, the heat flow across the line $ab$ is approximately

$$q_1 = k\,\Delta y\,(T_0 - T_1)/\Delta x \quad (28)$$

Similarly for each of the other sides of the square $abcd$. Since $\Delta x = \Delta y$, the total flow into the square is

$$q = k\,(4T_0 - T_1 - T_2 - T_3 - T_4) \quad (29)$$

Let $Q = q/k$. Then, if the heat source is located at the point $x,y$, it will be required that the steady state solution satisfy the relation

$$-T(x-\Delta x, y) - T(x+\Delta x, y) - T(x, y-\Delta y) - T(x, y+\Delta y) + 4T(x,y) = Q \quad (30)$$

while satisfying Equation (3) elsewhere. The iteration Equations (9) and (10) must then be modified at the heat source point in order to converge to this solution. They become, respectively,

*Implicit in* x *direction,*

$$-T^*(x-\Delta x, y) + (2+\alpha)\,T^*(x,y) - T^*(x+\Delta x, y) = Q + T(x, y-\Delta y) + (\alpha-2)\,T(x,y) + T(x, y+\Delta y) \quad (31)$$

*Implicit in* y *direction,*

$$-T^*(x, y-\Delta y) + (2+\alpha)\,T^*(x,y) - T^*(x, y+\Delta y) = Q + T(x-\Delta x, y) + (\alpha-2)\,T(x,y) + T(x+\Delta x, y) \quad (32)$$

Equations (31) and (32) may also be used as difference equations for unsteady state solutions.

To start the iteration, a temperature of zero was assigned to every point. As before, after each double step residuals were calculated at every interior point by

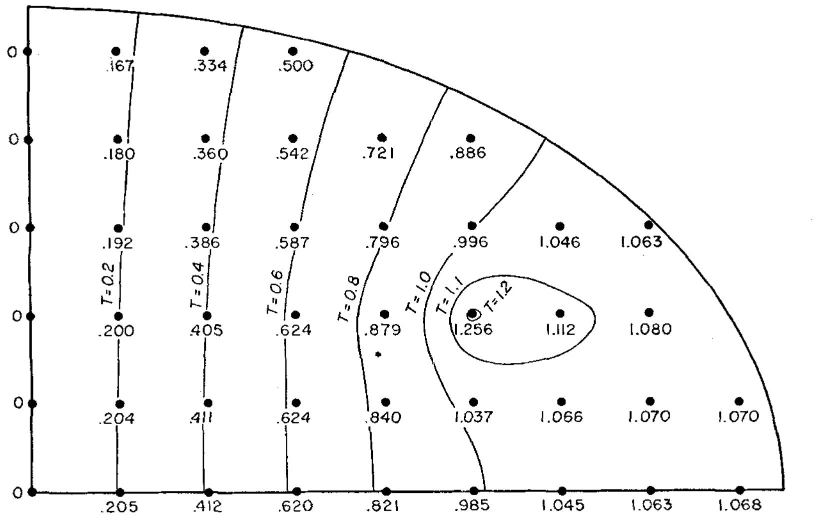


Fig. 11. Temperature distribution in an elliptical plate with a point heat source.

use of Equation (16), with the exception of the source point, where Equation (33) was used.

$$\delta_{35} = T_{34} + T_{36} + T_{25} + T_{45} - 4\,T_{35} + Q \qquad (33)$$

The values of $\alpha$ used were 4, 2, 1, 0.7, 0.4, 0.2, 0.1, 0.07, 0.04, 7, 4, 2, 1, 0.4, 0.1, 0.07, 0.04, 0.03, 10, 4, 1, 0.4, 0.1, 0.07, 7, 2, 0.7. A plot of log $\Sigma\delta^2$ vs. number of steps was *approximately straight*, with $\Sigma\delta^2$ decreasing from 1.0 to $5.23 \times 10^{-13}$ in twenty-seven double steps. The solution is presented in Figure 11.

## CHOICE OF ITERATION PARAMETER

For iteration to the steady-state solution of the heat-flow equation with a square boundary, a procedure has been suggested(7) for selection of a set of values of $\alpha$. Such a procedure has not been obtained, however, for other types of boundaries. The sets of values of $\alpha$ used to iterate these three examples by no means represent an optimum choice, and it appears that more experience with the alternating-direction implicit method will be required before an optimum procedure for choosing $\alpha$ can be found.

Some general statements can be made, however. In the first place, it is desirable to calculate the sum of squares of the residuals, in order to keep track of the progress of the iteration. Second, it is advantageous to use several different values for $\alpha$ rather than continuing to use the same value repeatedly. A satisfactory procedure appears to be the following: to start, an $\alpha$ between 4 and 10 is chosen; on each double step $\alpha$ is divided by 2 or 4, until the rate of decrease of $\Sigma\delta^2$ becomes small. The cycle is then repeated as many times as necessary to reduce the residuals to as low a level as desired. One or two such cycles should be sufficient for engineering accuracy in the solution.

## CONCLUSION

The chief advantage of the alternating-direction implicit method developed by Peaceman and Rachford(7) is that both unsteady and steady state problems in the flow of heat in two dimensions can be solved on a computing machine to a given degree of accuracy with less computing labor than that required by previously available numerical techniques. This reduction in labor is brought about by the fact that the new method is stable for any size of time increment and yet does not require iteration for solution of the simultaneous equations obtained.

Because it has not been possible to prove rigorously that the alternating-direction implicit method will work for nonrectangular boundary condition, it is necessary to rely upon actual numerical examples to determine the usefulness of the method for various engineering problems having realistic boundary conditions. The main purpose of this paper has been to present a few examples with a wide variety of boundary conditions in order to show the usefulness of the method for these problems. While it is not possible to say in advance that the new technique is applicable to any problem different from the ones described here, the success of the method with these problems, as well as the fact that no problems have yet been found where it is unsuccessful, are strong indications that the method may be applied to other heat flow or diffusion problems in two dimensions.

## NOTATION

$A,B,C,D$, = coefficients of simultaneous equations
$c$ = heat capacity, B.t.u./(lb.) (°F.)
$g$ = intermediate quantity in solution of simultaneous equations
$k$ = thermal conductivity, B.t.u./(hr.) (ft.) (°F.)
$L$ = basic length, ft.
$N$ = number of interior points of integration net
$n$ = distance in direction perpendicular to boundary
$Q$ = dimensionless rate of heat input, $q/k$
$q$ = rate of heat input, B.t.u./(hr.) (ft.)
$T$ = dimensionless temperature
$t$ = temperature, °R.
$w$ = intermediate quantity in solution of simultaneous equations
$X$ = dimensionless distance, $x/L$
$x$ = distance in horizontal direction, ft.
$Y$ = dimensionless distance, $y/L$
$y$ = distance in vertical direction, ft.
$\alpha$ = iteration parameter in steady state problems; $c\rho(\Delta x)^2/k\Delta\theta$ in unsteady state problems
$\Delta x,\Delta y,\Delta\theta$ = increment of $x,y$, and $\theta$, respectively
$\delta$ = residual
$\theta$ = time, hr.
$\rho$ = density, lb./cu.ft.
$\sigma$ = Stefan-Boltzmann constant

**Superscript**

$*$ = unknown values of $T$

**Subscripts**

$i$ = index number for simultaneous equations
1 = inside of pipe
2 = surroundings

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